Differential equations and logarithmic intertwining operators for strongly graded vertex algebras

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Abstract

We derive certain systems of differential equations for matrix elements of products and iterates of logarithmic intertwining operators among strongly graded generalized modules for a strongly graded conformal vertex algebra under suitable assumptions. Using these systems of differential equations, we verify the convergence and extension property needed in the logarithmic tensor category theory for such strongly graded generalized modules developed by Huang, Lepowsky and Zhang.

1 Introduction

In the present paper, we generalize the arguments in [H] and [HLZ] to prove that for a *strongly graded* conformal vertex algebra V, matrix elements of products and iterates of *logarithmic* intertwining operators among triples of strongly graded generalized V-modules under suitable assumptions satisfy certain systems of differential equations and that the prescribed singular points are regular. Using these differential equations, we verify the *convergence and extension property* needed in the theory of logarithmic tensor categories for strongly graded generalized V-modules in [HLZ]. Consequently, under certain assumptions on the strongly graded generalized modules for a strongly graded conformal vertex algebra V, we obtain a natural structure of braided tensor category on the category of strongly graded generalized V-modules using the main result of [HLZ].

The notion of strongly graded conformal vertex algebra and the notion of its strongly graded module were introduced in [HLZ] as natural concepts from which the theory of logarithmic tensor categories was developed. A strongly A-graded conformal vertex algebra V (respectively, a strongly $\tilde{A}$-graded V-module) is a vertex algebra (respectively, a V-module), with a weight-grading provided by a conformal vector in V (an $L(0)$-eigenspace decomposition), and with a second, compatible grading by an abelian group A (respectively, an abelian group $\tilde{A}$ containing A as its subgroup), satisfying certain grading restriction conditions. One important source of examples of strongly graded conformal vertex algebras and

modules comes from the vertex algebras and modules associated with *not necessarily positive definite* even lattices. In particular, the tensor products of vertex operator algebras and the vertex algebras associated with even lattices are strongly graded conformal vertex algebras (see [Y]). In [B1], Borcherds used the vertex algebra associated with the self-dual Lorentzian lattice of rank 2 and its tensor product with $V^\natural$ to construct the "Monster" Lie algebra.

It was proved in [H] that if every module W for a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ satisfies the C_1-*cofiniteness condition*, that is, dim $W/C_1(W) < \infty$, where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$, then matrix elements of products and iterates of intertwining operators among triples of V-modules satisfy certain systems of differential equations. Moreover, for prescribed singular points, there exist such systems of differential equations such that the prescribed singular points are regular. In Section 11 of [HLZ] (Part VII), using the same argument as in [H], certain systems of differential equations were derived for matrix elements of products and iterates of logarithmic intertwining operators among triples of generalized V-modules. In this paper, we prove similar, more general results for matrix elements of products and iterates of logarithmic intertwining operators among triples of strongly graded generalized modules for a strongly graded vertex algebra.

In the present paper, we generalize the C_1-cofiniteness condition for generalized modules for a vertex operator algebra to a C_1-cofiniteness condition for strongly graded generalized modules for a strongly graded vertex algebra. That is, every strongly graded generalized $\tilde{A}$-module W for a strongly A-graded vertex algebra V satisfies the condition dim $W^{(\beta)}/(C_1(W))^{(\beta)} < \infty$ where $C_1(W)$ is the subspace of W spanned by elements of the form $u_{-1}w$ for $u \in V_+ = \coprod_{n>0} V_{(n)}$ and $w \in W$, $W^{(\beta)}$ and $(C_1(W))^{(\beta)}$ are $\tilde{A}$-homogeneous subspace of W and $C_1(W)$ with $\tilde{A}$-grading β for $\beta \in \tilde{A}$. Furthermore, let V_0 be a strongly graded vertex subalgebra of V, the C_1-cofiniteness condition for W as a V_0-module implies the C_1-cofiniteness condition for W as a V-module. In particular, the case that W satisfies C_1-cofiniteness condition as a module for $V^{(0)}$—the A-homogeneous subspace of V with A-weight 0—is the same as the case that W satisfies C_1-cofiniteness condition as a vertex operator algebra module.

The key step in deriving systems of differential equations in [H] is to construct a finitely generated $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$-module that is a quotient module of the tensor product of R and a quadruple of modules for a vertex operator algebra. However, for a strongly graded conformal vertex algebra, the quotient module constructed in the same way is not finitely generated since there are infinitely many $\tilde{A}$-homogeneous subspaces in the strongly graded generalized modules. In order to obtain a finitely generated quotient module, we assume that fusion rules for triples of certain $\tilde{A}$-homogeneous subspaces of strongly graded generalized V-modules viewed as $V^{(0)}$-modules are zero for all but finitely many triples of such $\tilde{A}$-homogeneous subspaces.

Under the assumption on the fusion rules for triples of certain $\tilde{A}$-homogeneous subspaces and the C_1-cofiniteness condition for the strongly graded generalized modules, we construct a natural map from a finitely generated R-module to the set of matrix elements of products and iterates of logarithmic intertwining operators among triples of strongly graded generalized

V-modules. The images of certain elements under this map provide systems of differential equations for the matrix elements of products and iterates of logarithmic intertwining operators, as a consequence of the $L(-1)$-derivative property for the logarithmic intertwining operators. Moreover, for any prescribed singular point, we derive certain systems of differential equations such that this prescribed singular point is regular. Using these systems of differential equations, we verify the convergence and extension property needed in the construction of associativity isomorphism for the logarithmic tensor category structure developed in [HLZ]. Consequently, if all the assumptions mentioned above are satisfied, we obtain a braided tensor category structure on the category of strongly graded generalized V-modules.

The present paper is organized as follows: In section 2, we recall the definitions and some basic properties of strongly graded vertex algebras and their strongly graded generalized modules. The C_1-cofiniteness condition for strongly graded generalized modules is introduced in section 3 and the definitions of logarithmic intertwining operators among strongly graded generalized modules is recalled in section 4. The existence of systems of differential equations and the existence of systems with regular prescribed singular points are established in section 5 and 6, respectively. In section 7, we prove the convergence and extension property for products and iterates of logarithmic intertwining operators among strongly graded generalized modules for a strongly graded vertex algebra. Consequently, we obtain the braided tensor category structure on the category of strongly graded generalized modules generalizing the results in [HLZ].

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2 Strongly graded vertex algebras and their modules

In this section, we recall the basic definitions from [HLZ] (cf. [Y]).

Definition 2.1 A *conformal vertex algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

equipped with a linear map:

$$\begin{aligned} V &\to (\mathrm{End}\, V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}, \end{aligned}$$

and equipped also with two distinguished vectors: *vacuum vector* $\mathbf{1} \in V_{(0)}$ and *conformal vector* $\omega \in V_{(2)}$, satisfying the following conditions for $u, v \in V$:

- the *lower truncation condition*:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large;}$$

- the *vacuum property*:

$$Y(\mathbf{1}, x) = 1_V;$$

- the *creation property*:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v;$$

- the *Jacobi identity* (the main axiom):

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2);$$

- the *Virasoro algebra relations*:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2},$$

$$c \in \mathbb{C} \quad \text{(central charge of } V\text{)};$$

satisfying the $L(-1)$-*derivative property*:

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x);$$

and

$$L(0)v = nv = (\text{wt } v)v \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad v \in V_{(n)}.$$

This completes the definition of the notion of conformal vertex algebra. We will denote such a conformal vertex algebra by $(V, Y, \mathbf{1}, \omega)$.

Definition 2.2 Given a conformal vertex algebra $(V, Y, \mathbf{1}, \omega)$, a *module* for V is a $\mathbb{C}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} \tag{2.1}$$

equipped with a linear map

$$\begin{array}{rcl} V & \to & (\text{End } W)[[x, x^{-1}]] \\ v & \mapsto & Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{array}$$

such that the following conditions are satisfied:

- the lower truncation condition: for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for} \quad n \quad \text{sufficiently large;}$$

- the vacuum property:

$$Y(\mathbf{1}, x) = 1_W;$$

- the Jacobi identity for vertex operators on W: for $u, v \in V$,

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)$$
$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2);$$

- the Virasoro algebra relations on W with scalar c equal to the central charge of V:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c$$

  for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2};$$

  satisfying the $L(-1)$-derivative property

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x);$$

  and

$$(L(0) - n)w = 0 \quad \text{for} \quad n \in \mathbb{C} \quad \text{and} \quad w \in W_{(n)}. \tag{2.2}$$

This completes the definition of the notion of module for a conformal vertex algebra.

Definition 2.3 A generalized module for a conformal vertex algebra is defined in the same way as a module for a conformal vertex algebra except that in the grading (2.1), each space $W_{(n)}$ is replaced by $W_{[n]}$, where $W_{[n]}$ is the generalized $L(0)$-eigenspace corresponding to the generalized eigenvalue $n \in \mathbb{C}$; that is, (2.1) and (2.2) in the definition are replaced by

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

and

$$\text{for } n \in \mathbb{C} \text{ and } w \in W_{[n]}, \; (L(0) - n)^k w = 0, \text{ for } k \in \mathbb{N} \text{ sufficiently large,}$$

respectively. For $w \in W_{[n]}$, we still write wt $w = n$ for the generalized weight of w.

Definition 2.4 Let A be an abelian group. A conformal vertex algebra

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)}$$

is said to be *strongly graded with respect to* A (or *strongly* A*-graded*, or just *strongly graded* if the abelian group A is understood) if it is equipped with a second gradation, by A,

$$V = \coprod_{\alpha\in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n\in\mathbb{Z}} V^{(\alpha)}_{(n)}, \quad \text{where } V^{(\alpha)}_{(n)} = V_{(n)} \cap V^{(\alpha)} \quad \text{for any} \quad \alpha \in A;$$

for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$\begin{aligned}
&V^{(\alpha)}_{(n)} = 0 \quad \text{for} \quad n \quad \text{sufficiently negative;}\\
&\dim V^{(\alpha)}_{(n)} < \infty;\\
&\mathbf{1} \in V^{(0)}_{(0)}; \quad \omega \in V^{(0)}_{(2)};\\
&v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any} \quad v \in V^{(\alpha)},\ l \in \mathbb{Z}.
\end{aligned}$$

This completes the definition of the notion of strongly A-graded conformal vertex algebra.

For modules for a strongly graded algebra we will also have a second grading by an abelian group, and it is natural to allow this group to be larger than the second grading group A for the algebra. (Note that this already occurs for the *first* grading group, which is $\mathbb{Z}$ for algebras and $\mathbb{C}$ for modules.)

Definition 2.5 Let A be an abelian group and V a strongly A-graded conformal vertex algebra. Let $\tilde{A}$ be an abelian group containing A as a subgroup. A V-module (respectively, generalized V-module)

$$W = \coprod_{n\in\mathbb{C}} W_{(n)} \quad (\text{respectively, } \ W^{(\beta)} = \coprod_{n\in\mathbb{C}} W_{[n]})$$

is said to be *strongly graded with respect to* $\tilde{A}$ (or *strongly* $\tilde{A}$*-graded*, or just *strongly graded*) if the abelian group $\tilde{A}$ is understood) if it is equipped with a second gradation, by $\tilde{A}$,

$$W = \coprod_{\beta\in\tilde{A}} W^{(\beta)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n\in\mathbb{C}} W^{(\beta)}_{(n)}, \quad \text{where} \quad W^{(\beta)}_{(n)} = W_{(n)} \cap W^{(\beta)}$$

$$(\text{respectively, } W^{(\beta)} = \coprod_{n\in\mathbb{C}} W^{(\beta)}_{[n]}, \text{ where } W^{(\beta)}_{[n]} = W_{[n]} \cap W^{(\beta)});$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{(n+k)} = 0 \ \ (\text{respectively, } W^{(\beta)}_{[n+k]} = 0) \quad \text{for } k \in \mathbb{Z} \text{ sufficiently negative;} \tag{2.3}$$
$$\dim W^{(\beta)}_{(n)} < \infty \ \ (\text{respectively, } \dim W^{(\beta)}_{[n]} < \infty);$$
$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any } v \in V^{(\alpha)},\ l \in \mathbb{Z}.$$

A strongly $\tilde{A}$-graded (generalized) V-module W is said to be *lower bounded* if instead of (2.3), it satisfies the stronger condition that for any $\beta \in \tilde{A}$,

$$W^{(\beta)}_{(n)} = 0 \ \ (\text{respectively, } W^{(\beta)}_{[n]} = 0) \ \ \text{for } n \in \mathbb{C} \text{ and } \Re(n) \text{ sufficiently negative.}$$

This completes the definition of the notion of strongly $\tilde{A}$-graded generalized module for a strongly A-graded conformal vertex algebra.

Remark 2.6 In the strongly graded case, subalgebras (submodules) are vertex subalgebras (submodules) that are strongly graded; algebra and module homomorphisms are of course understood to preserve the grading by A or $\tilde{A}$.

With the strong gradedness condition on a (generalized) module, we can now define the corresponding notion of contragredient module.

Definition 2.7 Let $W = \coprod_{\beta\in\tilde{A}, n\in\mathbb{C}} W^{(\beta)}_{[n]}$ be a strongly $\tilde{A}$-graded generalized module for a strongly A-graded conformal vertex algebra. For each $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, let us identify $(W^{(\beta)}_{[n]})^*$ with the subspace of W^* consisting of the linear function on W vanishing on each $W^{(\gamma)}_{[n]}$ with $\gamma \neq \beta$ or $m \neq n$. We define W' to be the $(\tilde{A} \times \mathbb{C})$-graded vector subspaces of W^* given by

$$W' = \coprod_{\beta\in\tilde{A}, n\in\mathbb{C}} (W')^{(\beta)}_{[n]}, \ \ \text{where } \ (W')^{(\beta)}_{[n]} = (W^{(-\beta)}_{[n]})^*.$$

The *adjoint vertex operators* $Y'(v, z)$ $(v \in V)$ on W' is defined in the same way as vertex operator algebra in section 5.2 in [FHL] (see Section 2 of [HLZ]). The pair (W', Y') carries a strongly graded module structure as follow:

Proposition 2.8 *Let $\tilde{A}$ be an abelian group containing A as a subgroup and V a strongly A-graded conformal vertex algebra. Let (W, Y) be a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module). Then the pair (W', Y') carries a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module) structure. If W is lower bounded, so is W'.*

Definition 2.9 The pair (W', Y') is called the *contragredient module* of (W, Y).

Example 2.10 Note that the notion of conformal vertex algebra strongly graded with respect to the trivial group is exactly the notion of vertex operator algebra. Let V be a vertex operator algebra, viewed (equivalently) as a conformal vertex algebra strongly graded with respect to the trivial group. Then the V-modules that are strongly graded with respect to the trivial group (in the sense of Definition 2.5) are exactly the ($\mathbb{C}$-graded) modules for V as a vertex operator algebra, with the grading restrictions as follows: For $n \in \mathbb{C}$,

$$W_{(n+k)} = 0 \quad \text{for} \;\; k \in \mathbb{Z} \;\; \text{sufficiently negative}$$

and

$$\dim W_{(n)} < \infty.$$

Example 2.11 An important source of examples of strongly graded conformal vertex algebras and modules comes from the vertex algebras and modules associated with even lattices. We recall the following construction from [FLM]. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite, such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathfrak{h}$ is a vector space with a nonsingular bilinear form $\langle \cdot, \cdot \rangle$, extended from L. We form a Heisenberg algebra

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z},\; n \neq 0} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.$$

Let $(\widehat{L}, ^-)$ be a central extension of L by a finite cyclic group $\langle \kappa \mid \kappa^s = 1 \rangle$. Fix a primitive sth root of unity, say ω, and define the faithful character

$$\chi : \langle \kappa \rangle \to \mathbb{C}^*$$

by the condition

$$\chi(\kappa) = \omega.$$

Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle \kappa \rangle$-module on which $\langle \kappa \rangle$ acts according to χ:

$$\kappa \cdot 1 = \omega,$$

and denote by $\mathbb{C}\{L\}$ the induced $\widehat{L}$-module

$$\mathbb{C}\{L\} = \mathrm{Ind}_{\langle \kappa \rangle}^{\widehat{L}} \mathbb{C}_\chi = \mathbb{C}[\widehat{L}] \otimes_{\mathbb{C}[\langle \kappa \rangle]} \mathbb{C}_\chi.$$

Then

$$V_L = S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L\}$$

has a natural structure of conformal vertex algebra; see [B1] and Chapter 8 of [FLM]. For $\alpha \in L$, choose an $a \in \widehat{L}$ such that $\bar{a} = \alpha$. Define

$$\iota(a) = a \otimes 1 \in \mathbb{C}\{L\}$$

and

$$V_L^{(\alpha)} = \text{span}\,\{h_1(-n_1)\cdots h_k(-n_k)\otimes\iota(a)\},$$

where $h_1,\ldots,h_k\in\mathfrak{h}$, $n_1,\ldots,n_k>0$, and where $h(n)$ is the natural operator associated with $h\otimes t^n$ via the $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module structure of V_L. Then V_L is equipped with a natural second grading given by L itself. Also for $n\in\mathbb{Z}$, we have

$$(V_L)_{(n)}^{(\alpha)} = \text{span}\,\{h_1(-n_1)\cdots h_k(-n_k)\otimes\iota(a)|\ \bar{a}=\alpha, \sum_{i=1}^{k} n_i + \frac{1}{2}\langle\alpha,\alpha\rangle = n\},$$

making V_L a strongly L-graded conformal vertex algebra in the sense of definition 2.4. When the form $\langle\cdot,\cdot\rangle$ on L is also positive definite, then V_L is a vertex operator algebra, that is, as in example 2.10, V_L is a strongly A-graded conformal vertex algebra for A the trivial group. In general, a conformal vertex algebra may be strongly graded for several choinces of A.

Any sublattice M of the "dual lattice" L° of L containing L gives rise to a strongly M-graded module for the strongly L-graded conformal vertex algebra (see Chapter 8 of [FLM]; cf. [LL]). In fact, any irreducible (generalized) V_L-module is equivalent to a V_L-module of the form $V_{L+\beta}\subset V_{L^\circ}$ for some $\beta\in L^\circ$ and any (generalized) V_L-module W is equivalent to a direct sum of irreducible V_L-modules. i.e.,

$$W = \coprod_{\gamma_i\in L^\circ,\ i=1,\ldots,n} V_{\gamma_i+L},$$

where γ_i's are arbitrary elements of L°, and $n\in\mathbb{N}$ (see [D], [DLM]; cf. [LL]).

Definition 2.12 Let V be a strongly A-graded conformal vertex algebra. The subspaces $V_{(n)}^{(\alpha)}$ for $n\in\mathbb{Z}$, $\alpha\in A$ are called the *doubly homogeneous subspaces* of V. The elements in $V_{(n)}^{(\alpha)}$ are called *doubly homogeneous* elements. Similar definitions can be used for $W_{(n)}^{(\beta)}$ (respectively, $W_{[n]}^{(\beta)}$) in the strongly graded (generalized) module W.

Notation 2.13 Let v be a doubly homogeneous element of V. Let wt v_n, $n\in\mathbb{Z}$, refer to the weight of v_n as an operator acting on W, and let A-wt v_n refer to the A-weight of v_n on W. Similarly, let w be a doubly homogeneous element of W. We use wt w to denote the weight of w and $\tilde{A}$-wt w to denote the $\tilde{A}$-grading of w.

Lemma 2.14 *Let $v\in V_{(n)}^{(\alpha)}$, for $n\in\mathbb{Z}$, $\alpha\in A$. Then for $m\in\mathbb{Z}$,* wt $v_m = n-m-1$ *and* A-wt $v_m=\alpha$.

Proof. The first equation is standard from the theory of graded conformal vertex algebras and the second follows easily from the definitions. ∎

3 C_1-cofiniteness condition

In this section, we will let V denote a strongly A-graded conformal vertex algebra and let W denote a strongly $\tilde{A}$-graded lower bounded (generalized) V-module, where A, $\tilde{A}$ are abelian groups such that $A \subset \tilde{A}$.

In the following definition, we generalize the C_1-cofiniteness condition for the (generalized) modules for a vertex operator algebra to a C_1-cofiniteness condition for the strongly graded (generalized) modules for a strongly graded conformal vertex algebra.

Definition 3.1 Let $C_1(W)$ be the subspace of W spanned by elements of the form $u_{-1}w$ for

$$u \in V_+ = \coprod_{n>0} V_{(n)}$$

and $w \in W$. The $\tilde{A}$-grading on W induces a $\tilde{A}$-grading on $W/C_1(W)$ with

$$(W/C_1(W))^{(\beta)} = W^{(\beta)}/(C_1(W))^{(\beta)}.$$

If $\dim\,(W/C_1(W))^{(\beta)} < \infty$ for $\beta \in \tilde{A}$, we say that W is C_1-*cofinite* or W satisfies the C_1-*cofiniteness condition*.

Remark 3.2 Let V_0 be a conformal vertex subalgebra of V strongly graded with respect to $A_0 \subset A$. We can also define C_1-cofiniteness condition for W as a strongly graded (generalized) V_0-module. If W is C_1-cofinite as a strongly graded (generalized) V_0-module. Then W is C_1-cofinite as a strongly graded (generalized) V-module.

Example 3.3 Let V_L be the conformal vertex algebra associated with a nondegenerate even lattice L and let W be a (generalized) V_L-module as in Example 2.11. Then the strongly graded (generalized) V_L-module W satisfies the C_1-cofiniteness condition as a $V_L^{(0)}$-module. Thus W is also C_1-cofinite as a strongly graded V_L-module.

4 Logarithmic intertwining operators

Throughout this paper, we shall use $x, x_0, x_1, x_2, \ldots$ to denote commuting formal variables and $z, z_0, z_1, z_2, \ldots$ to denote complex variables or complex numbers. We first recall the following definitions from [HLZ].

Definition 4.1 Let (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) be generalized modules for a conformal vertex algebra V. A *logarithmic intertwining operator of type* $\binom{W_3}{W_1\,W_2}$ is a linear map

$$\mathcal{Y}(\cdot, x)\cdot : W_1 \otimes W_2 \to W_3[\log x]\{x\}, \tag{4.1}$$

or equivalently,

$$w_{(1)} \otimes w_{(2)} \mapsto \mathcal{Y}(w_{(1)}, x)w_{(2)} = \sum_{n\in\mathbb{C}} \sum_{k\in\mathbb{N}} w_{(1)}{}^{\mathcal{Y}}_{n;\,k} w_{(2)} x^{-n-1}(\log x)^k \in W_3[\log x]\{x\} \tag{4.2}$$

for all $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that the following conditions are satisfied: the *lower truncation condition*: for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $n \in \mathbb{C}$,

$$w_{(1)}{}^{\mathcal{Y}}_{n+m;\,k} w_{(2)} = 0 \quad \text{for} \;\; m \in \mathbb{N} \;\; \text{sufficiently large, independently of } k; \tag{4.3}$$

the *Jacobi identity*:

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)} \\ &\qquad - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)} \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \end{aligned} \tag{4.4}$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$ (note that the first term on the left-hand side is meaningful because of (4.3)); the $L(-1)$-*derivative property:* for any $w_{(1)} \in W_1$,

$$\mathcal{Y}(L(-1)w_{(1)}, x) = \frac{d}{dx}\mathcal{Y}(w_{(1)}, x). \tag{4.5}$$

Definition 4.2 In the setting of Definition 4.1, suppose in addition that V and W_1, W_2 and W_3 are strongly graded. A logarithmic intertwining operator $\mathcal{Y}$ as in Definition 4.1 is a *grading-compatible logarithmic intertwining operator* if for $\beta, \gamma \in \tilde{A}$ and $w_1 \in W_1^{(\beta)}$, $w_2 \in W_2^{(\gamma)}$, $n \in \mathbb{C}$ and $k \in \mathbb{N}$, we have

$$(w_1)_{n;k} w_2 \in W_3^{(\beta+\gamma)}.$$

Definition 4.3 In the setting of Definition 4.2, the grading-compatible logarithmic intertwining operators of a fixed type $\binom{W_3}{W_1\,W_2}$ form a vector space, which we denote by $\mathcal{V}^{W_3}_{W_1 W_2}$. We call the dimension of $\mathcal{V}^{W_3}_{W_1 W_2}$ the *fusion rule* for W_1, W_2 and W_3 and denote it by $N^{W_3}_{W_1 W_2}$.

We shall use the following two sets in the next section: For $\beta_i \in \tilde{A}$, $i = 1, 2, 3$, set

$$\tilde{I}^{(\beta_1,\beta_2,\beta_3)} = (\beta_1 + A_0) \times (\beta_2 + A_0) \times (\beta_3 + A_0).$$

For any strongly $\tilde{A}$-graded generalized V-modules W_i $(i = 0, 1, \ldots, 4)$ and any logarithmic intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of type $\binom{W_0'}{W_1\,W_4}$ and $\binom{W_4}{W_2\,W_3}$, respectively, set

$$I^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2} = \left\{ (\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)} \;\middle|\; \begin{array}{c} \text{if there exist} \;\; w_i \in W_i^{(\widetilde{\beta}_i)} \;(i = 1, 2, 3) \;\text{such that} \\ \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3 \neq 0 \end{array} \right\}.$$

For brevity, we will use $I^{(\beta_1,\beta_2,\beta_3)}$ to denote the set $I^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$ in the rest of this paper.

Lemma 4.4 *Let V be a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to $A_0 \subset A$. Suppose that every strongly graded V-module satisfies C_1-cofiniteness condition as a V_0-module. Also suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})}M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Then the set $I^{(\beta_1,\beta_2,\beta_3)}$ defined above is a finite set.

Proof. Since for the triple of strongly graded generalized modules (W_1, W_2, W_3), the fusion rules $N^{W_3^{(\tilde{\beta}_1+\tilde{\beta}_2)}}_{W_1^{(\widetilde{\beta_1})}W_2^{(\widetilde{\beta_2})}} \neq 0$ for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$, the logarithmic intertwining operator $\mathcal{Y}_2(w_2, x_2)w_3$, where $w_2 \in W_2^{(\widetilde{\beta_2})}$ and $w_3 \in W_3^{(\widetilde{\beta_3})}$, have to be 0 except for finitely many pairs $(\widetilde{\beta_2}, \widetilde{\beta_3}) \in (\beta_2 + A_0) \times (\beta_3 + A_0)$, and then there are only finitely many triples $(\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ such that the products of logarithmic intertwining operators

$$\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3 \neq 0,$$

where $w_1 \in W_1^{(\widetilde{\beta_1})}$, $w_2 \in W_2^{(\widetilde{\beta_2})}$ and $w_3 \in W_3^{(\widetilde{\beta_3})}$. Thus the set $I^{(\beta_1,\beta_2,\beta_3)}$ is a finite set. ∎

Remark 4.5 In the case that A_0 is a finite subgroup of A, the assumption in Lemma 4.4 holds automatically.

5 Differential equations

In the rest of this paper, we assume that V is a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to $A_0 \subset A$, and assume that every strongly graded (generalized) V-module is $\mathbb{R}$-graded, lower bounded and satisfies C_1-cofiniteness condition as a V_0-module.

Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, \ldots, 4$ and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators of type $\binom{W_0'}{W_1\, W_4}$ and $\binom{W_4}{W_2\, W_3}$, respectively. Let $\tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ and $I^{(\beta_1,\beta_2,\beta_3)}$ be the two sets defined in the previous section.

Let $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$, β_1, β_2 and β_3 be three fixed elements in $\tilde{A}$. Set

$$\widetilde{T}^{(\beta_1,\beta_2,\beta_3)} = \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3})\in\tilde{I}^{(\beta_1,\beta_2,\beta_3)}} R \otimes W_0^{(\widetilde{\beta_1}+\widetilde{\beta_2}+\widetilde{\beta_3})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}$$

and

$$T^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2} = \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3})\in I^{(\beta_1,\beta_2,\beta_3)}} R \otimes W_0^{(\widetilde{\beta_1}+\widetilde{\beta_2}+\widetilde{\beta_3})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}.$$

Then $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ and $T^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$ have natural R-module structures. For convenience, in the rest of this paper, we will use $T^{(\beta_1,\beta_2,\beta_3)}$ to denote $T^{(\beta_1,\beta_2,\beta_3)}_{\mathcal{Y}_1,\mathcal{Y}_2}$.

For simplicity, we shall omit one tensor symbol to write $f(z_1,z_2)\otimes w_0\otimes w_1\otimes w_2\otimes w_3$ as $f(z_1,z_2)w_0\otimes w_1\otimes w_2\otimes w_3$ in $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ and $T^{(\beta_1,\beta_2,\beta_3)}$. For a strongly $\tilde{A}$-graded generalized V-module W, let (W',Y') be the contragredient module of W (recall definition 2.9). In particular, for $u\in V$ and $n\in\mathbb{Z}$, we have the operators u_n on W'. Let $u_n^*: W\to W$ be the adjoint of $u_n: W'\to W'$. Note that since wt $u_n=$ wt $u-n-1$, we have wt $u_n^*=-$wt $u+n+1$. Also, A-wt $u_n^*=-(A\text{-wt } u_n)$.

Let $(\widetilde{\beta}_1,\widetilde{\beta}_2,\widetilde{\beta}_3)\in\tilde{I}^{(\beta_1,\beta_2,\beta_3)}$ and let $\widetilde{\beta}_0=\widetilde{\beta}_1+\widetilde{\beta}_2+\widetilde{\beta}_3$. For $u\in(V_0)_+$ and $w_i\in W_i^{(\widetilde{\beta}_i)}$ $(i=0,1,2,3)$, let $J^{(\beta_1,\beta_2,\beta_3)}$ be the submodule of $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$ generated by elements of the form

$$\begin{aligned}
&\mathcal{A}(u,w_0,w_1,w_2,w_3)\\
&\quad=\sum_{k\geq 0}\binom{-1}{k}(-z_1)^k u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3-w_0\otimes u_{-1}w_1\otimes w_2\otimes w_3\\
&\qquad-\sum_{k\geq 0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}w_0\otimes w_1\otimes u_kw_2\otimes w_3\\
&\qquad-\sum_{k\geq 0}\binom{-1}{k}(-z_1)^{-1-k}w_0\otimes w_1\otimes w_2\otimes u_kw_3,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}(u,w_0,w_1,w_2,w_3)\\
&=\sum_{k\geq 0}\binom{-1}{k}(-z_2)^k u^*_{-1-k}w_0\otimes w_1\otimes w_2\otimes w_3\\
&\quad-\sum_{k\geq 0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}w_0\otimes u_kw_1\otimes w_2\otimes w_3-w_0\otimes w_1\otimes u_{-1}w_2\otimes w_3\\
&\quad-\sum_{k\geq 0}\binom{-1}{k}(-z_2)^{-1-k}w_0\otimes w_1\otimes w_2\otimes u_kw_3,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{C}(u,w_0,w_1,w_2,w_3)\\
&\quad=u^*_{-1}w_0\otimes w_1\otimes w_2\otimes w_3-\sum_{k\geq 0}\binom{-1}{k}z_1^{-1-k}w_0\otimes u_kw_1\otimes w_2\otimes w_3\\
&\qquad-\sum_{k\geq 0}\binom{-1}{k}z_2^{-1-k}w_0\otimes w_1\otimes u_kw_2\otimes w_3-w_0\otimes w_1\otimes w_2\otimes u_{-1}w_3,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}(u, w_0, w_1, w_2, w_3) \\
&\quad = u_{-1}w_0 \otimes w_1 \otimes w_2 \otimes w_3 \\
&\qquad - \sum_{k\geq 0} \left(\begin{array}{c} -1 \\ k \end{array} \right) z_1^{k+1} w_0 \otimes e^{z_1^{-1}L(1)}(-z_1^2)^{L(0)} u_k (-z_1^{-2})^{L(0)} e^{-z_1^{-1}L(1)} w_1 \otimes w_2 \otimes w_3 \\
&\qquad - \sum_{k\geq 0} \left(\begin{array}{c} -1 \\ k \end{array} \right) z_2^{k+1} w_0 \otimes w_1 \otimes e^{z_2^{-1}L(1)}(-z_2^2)^{L(0)} u_k (-z_2^{-2})^{L(0)} e^{-z_2^{-1}L(1)} w_2 \otimes w_3 \\
&\qquad - w_0 \otimes w_1 \otimes w_2 \otimes u_{-1}^* w_3.
\end{aligned}$$

We shall also need a submodule $S_{\mathcal{Y}_1,\mathcal{Y}_2}^{(\beta_1,\beta_2,\beta_3)}$ of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$ generated by elements of the form

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3$$

for $w_i \in W_i^{(\widetilde{\beta}_i)} (i = 0, 1, 2, 3)$, $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in \tilde{I}^{(\beta_1,\beta_2,\beta_3)} \setminus I^{(\beta_1,\beta_2,\beta_3)}$. For simplicity, we denote $S_{\mathcal{Y}_1,\mathcal{Y}_2}^{(\beta_1,\beta_2,\beta_3)}$ by $S^{(\beta_1,\beta_2,\beta_3)}$.

Lemma 5.1 *Let $\beta_i \in \tilde{A}$. Then*

$$\widetilde{T}^{(\beta_1,\beta_2,\beta_3)} = T^{(\beta_1,\beta_2,\beta_3)} \oplus S^{(\beta_1,\beta_2,\beta_3)}.$$

We shall find an R-submodule of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$ such that its complement in $T^{(\beta_1,\beta_2,\beta_3)}$ is finitely generated. For this purpose, we use the following R-submodule of $\tilde{T}^{(\beta_1,\beta_2,\beta_3)}$:

$$\tilde{J}^{(\beta_1,\beta_2,\beta_3)} = J^{(\beta_1,\beta_2,\beta_3)} \oplus S^{(\beta_1,\beta_2,\beta_3)}.$$

For $r \in R$, we can define the R-submodules $T_{(r)}^{(\beta_1,\beta_2,\beta_3)}$, $F_r(T^{(\beta_1,\beta_2,\beta_3)})$ and $F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ as in [H]. Note that $F_r(T^{(\beta_1,\beta_2,\beta_3)})$ is a finitely generated R-module since $I^{(\beta_1,\beta_2,\beta_3)}$ is a finite set by Lemma 4.4.

Proposition 5.2 *Let W_i be strongly $\tilde{A}$-graded generalized V-modules and let $\beta_i \in \tilde{A}$ for $i = 0, 1, 2, 3$. Then there exists $M \in \mathbb{Z}$ such that for any $r \in \mathbb{R}$, $F_r(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. In particular, $T^{(\beta_1,\beta_2,\beta_3)} \subset \tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)})$.*

Proof. For $\widetilde{\beta}_i \in \tilde{A}$, let $\widetilde{\beta}_0$ denote $\widetilde{\beta}_1 + \widetilde{\beta}_2 + \widetilde{\beta}_3$ and let $(C_1(W_i))^{(\widetilde{\beta}_i)}$ be the subspace of W_i spanned by elements of the form $u_{-1}w_i \in W_i^{(\widetilde{\beta}_i)}$, where

$$u \in (V_0)_+ = \coprod_{n>0} (V_0)_{(n)}.$$

Since $\dim W_i^{(\widetilde{\beta_i})}/(C_1(W_i))^{(\widetilde{\beta_i})} < \infty$ for $i = 0, 1, 2, 3$, there exists $M \in \mathbb{Z}$ such that

$$\begin{aligned}
\coprod_{n>M} T_{(n)}^{(\beta_1,\beta_2,\beta_3)} \subset \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}} \quad & R((C_1(W_0))^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\
+ \quad & R(W_0^{(\widetilde{\beta_0})} \otimes (C_1(W_1))^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\
+ \quad & R(W_0^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes (C_1(W_2))^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}) \\
+ \quad & R(W_0^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes (C_1(W_3))^{(\widetilde{\beta_3})}).
\end{aligned} \tag{5.1}$$

We use induction on $r \in \mathbb{R}$. If r is equal to M, $F_M(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_M(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Now we assume that $F_r(T^{(\beta_1,\beta_2,\beta_3)}) \subset F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$ for $r < s$ where $s > M$. We want to show that any homogeneous element of $T_{(s)}^{(\beta_1,\beta_2,\beta_3)}$ can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Since $s > M$, by (5.1), any element of $T_{(s)}^{(\beta_1,\beta_2,\beta_3)}$ is an element of the right hand side of (5.1). We shall discuss only the case that this element is in $R(W_0^{(\widetilde{\beta_0})} \otimes (C_1(W_1))^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})})$; the other cases are completely similar.

We need only discuss elements of the form $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$, where $w_i \in W_i^{(\widetilde{\beta_i})}$ for $i = 0, 2, 3$, $u_{-1}w_1 \in (C_1(W_1))^{(\widetilde{\beta_1})}$ and $u \in (V_0)_+$. We see from Lemma 5.1 that the elements $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ and $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ for $k \geq 0$ are either in $S^{(\beta_1,\beta_2,\beta_3)}$ or in $T^{(\beta_1,\beta_2,\beta_3)}$. By assumption, the weight of $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ is s, then the weight of $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ and $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ for $k \geq 0$, are all less than s. Thus these elements either lie in $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ or in $F_{s-1}(T^{(\beta_1,\beta_2,\beta_3)})$. Also, since $\mathcal{A}(u, w_0, w_1, w_2, w_3) \in F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$, we see that

$$\begin{aligned}
& w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3 \\
& \quad = \mathcal{A}(u, w_0, w_1, w_2, w_3) + \sum_{k\geq 0} \binom{-1}{k} (-z_1)^k u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3 \\
& \quad\quad - \sum_{k\geq 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \\
& \quad\quad - \sum_{k\geq 0} \binom{-1}{k} (-z_1)^{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3
\end{aligned}$$

can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and elements of $F_{s-1}(T^{(\beta_1,\beta_2,\beta_3)})$. Thus by the induction assumption, the element $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$.

Now we have

$$\begin{aligned} T^{(\beta_1,\beta_2,\beta_3)} &= \coprod_{r\in\mathbb{R}} F_r(T^{(\beta_1,\beta_2,\beta_3)}) \\ &\subset \coprod_{r\in\mathbb{R}} F_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)}) \\ &= \tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)}). \end{aligned}$$

∎

We immediately obtain the following:

Corollary 5.3 *The quotient R-module $T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ is finitely generated.*

Proof. We have the following R-module isomorphism:

$$T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)} \simeq T^{(\beta_1,\beta_2,\beta_3)} + \tilde{J}^{(\beta_1,\beta_2,\beta_3)}/\tilde{J}^{(\beta_1,\beta_2,\beta_3)}.$$

By the previous Proposition, the R-module $T^{(\beta_1,\beta_2,\beta_3)} + \tilde{J}^{(\beta_1,\beta_2,\beta_3)}/\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ is a submodule of

$$\tilde{J}^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)})/\tilde{J}^{(\beta_1,\beta_2,\beta_3)} \simeq F_M(T^{(\beta_1,\beta_2,\beta_3)})/F_M(T^{(\beta_1,\beta_2,\beta_3)}) \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)},$$

which is finitely generated. ∎

For an element $\mathcal{W} \in T^{(\beta_1,\beta_2,\beta_3)}$, we shall use $[\mathcal{W}]$ to denote the equivalence class in $T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ containing $\mathcal{W}$. We also have:

Corollary 5.4 *Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, 2, 3$. For any $\tilde{A}$-homogeneous elements $w_i \in W_i$ $(i = 0, 1, 2, 3)$, let M_1 and M_2 be the R-submodules of $T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ generated by $[w_0 \otimes L(-1)^j w_1 \otimes w_2 \otimes w_3]$, $j \geq 0$, and by $[w_0 \otimes w_1 \otimes L(-1)^j w_2 \otimes w_3]$, $j \geq 0$, respectively. Then M_1, M_2 are finitely generated. In particular, for any $\tilde{A}$-homogeneous elements $w_i \in W_i$ $(i = 0, 1, 2, 3)$, there exist $a_k(z_1, z_2)$, $b_l(z_1, z_2) \in R$ for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ such that*

$$\begin{aligned} &[w_0 \otimes L(-1)^m w_1 \otimes w_2 \otimes w_3] + a_1(z_1, z_2)[w_0 \otimes L(-1)^{m-1} w_1 \otimes w_2 \otimes w_3] \\ &\qquad + \cdots + a_m(z_1, z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0, \end{aligned} \tag{5.2}$$

$$\begin{aligned} &[w_0 \otimes w_1 \otimes L(-1)^n w_2 \otimes w_3] + b_1(z_1, z_2)[w_0 \otimes w_1 \otimes L(-1)^{n-1} w_2 \otimes w_3] \\ &\qquad + \cdots + b_n(z_1, z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0. \end{aligned} \tag{5.3}$$

Now we establish the existence of systems of differential equations:

Theorem 5.5 *Let V be a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to $A_0 \subset A$. Suppose that every strongly graded V-module satisfies C_1-cofiniteness condition as a V_0-module. Also suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})}\, M_2^{(\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, 1, 2, 3, 4$ and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators of type $\binom{W_0'}{W_1\, W_4}$, $\binom{W_4}{W_2\, W_3}$. Then for any $\tilde{A}$-homogeneous elements $w_i \in W_i$ ($i = 0, 1, 2, 3$), there exist

$$a_k(z_1, z_2), b_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ such that the series

$$\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle, \tag{5.4}$$

satisfying the expansions of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_1^m} + a_1(z_1, z_2)\frac{\partial^{m-1}\varphi}{\partial z_1^{m-1}} + \cdots + a_m(z_1, z_2)\varphi = 0, \tag{5.5}$$

$$\frac{\partial^n \varphi}{\partial z_2^n} + b_1(z_1, z_2)\frac{\partial^{n-1}\varphi}{\partial z_2^{n-1}} + \cdots + b_n(z_1, z_2)\varphi = 0 \tag{5.6}$$

in the region $|z_1| > |z_2| > 0$.

Proof. The proof is similar to the proof of Theorem 1.4 in [H] except the difference on the R-module $\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$. We sketch the proof as follows:

Let $\Delta = \mathrm{wt}\ w_0 - \mathrm{wt}\ w_1 - \mathrm{wt}\ w_2 - \mathrm{wt}\ w_3$. For $(\widetilde{\beta_1}, \widetilde{\beta_2}, \widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}$, let $\widetilde{\beta_0} = \widetilde{\beta_1} + \widetilde{\beta_2} + \widetilde{\beta_3}$. Let $\mathbb{C}(\{x\})$ be the space of all series of the form $\sum_{n\in\mathbb{R}} a_n x^n$ for $n \in \mathbb{R}$ such that $a_n = 0$ when the real part of n is sufficiently negative.

Consider the map

$$\phi_{\mathcal{Y}_1,\mathcal{Y}_2} : T^{(\beta_1,\beta_2,\beta_3)} \longrightarrow z_1^{\Delta}\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}]$$

defined by

$$\begin{aligned} &\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(f(z_1, z_2)w_0 \otimes w_1 \otimes w_2 \otimes w_3) \\ &= \iota_{|z_1|>|z_2|>0}(f(z_1, z_2))\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle, \end{aligned}$$

where

$$\iota_{|z_1|>|z_2|>0} : R \longrightarrow \mathbb{C}[[z_2/z_1]][z_1^{\pm 1}, z_2^{\pm 1}]$$

is the map expanding elements of R as series in the regions $|z_1| > |z_2| > 0$.

Using the Jacobi identity for the logarithmic intertwining operators, we have that elements of $J^{(\beta_1,\beta_2,\beta_3)}$ are in the kernel of $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$. The elements of $S^{(\beta_1,\beta_2,\beta_3)}$ are in the kernel by the construction of the set $I^{(\beta_1,\beta_2,\beta_3)}$. From Lemma 5.1, we have

$$\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) = 0.$$

Thus the map $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$ induces a map

$$\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2} : T^{(\beta_1,\beta_2,\beta_3)}/T^{(\beta_1,\beta_2,\beta_3)} \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)} \longrightarrow z_1^{\Delta}\mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}].$$

Applying $\bar{\phi}_{\mathcal{Y}_1,\mathcal{Y}_2}$ to (5.2) and (5.3) and then use the $L(-1)$-derivative property for logarithmic intertwining operators, we see that (5.4) indeed satisfies the expansions of the system of differential equations in the regions $|z_1| > |z_2| > 0$. ∎

Remark 5.6 Note that in the theorems above, $a_k(z_1; z_2)$ for $k = 1, \dots, m-1$ and $b_l(z_1; z_2)$ for $l = 1, \dots, l-1$, and consequently the corresponding system, depend on the logarithmic intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$.

The following result can be proved in the same method, so we omit the proof.

Theorem 5.7 *Let V be a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to $A_0 \subset A$. Suppose that every strongly graded V-module satisfies C_1-cofiniteness condition as a V_0-module. Also suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rules*

$$N_{M_1^{(\widetilde{\beta_1})} M_2^{(\widetilde{\beta_2})}}^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i be strongly $\tilde{A}$-graded generalized V-modules for $i = 0, \dots, n+1$. For any generalized V-modules $\widetilde{W_1}, \dots, \widetilde{W_{n-1}}$, let

$$\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

be logarithmic intertwining operators of types

$$\binom{W_0}{W_1\, \widetilde{W_1}}, \binom{\widetilde{W_1}}{W_2\, \widetilde{W_2}}, \dots, \binom{\widetilde{W_{n-2}}}{W_{n-1}\, \widetilde{W_{n-1}}}, \binom{\widetilde{W_{n-1}}}{W_n\, W_{n+1}},$$

respectively. Then for any $\tilde{A}$-homogeneous elements $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \dots, w_{(n+1)} \in W_{n+1}$, there exist

$$a_{k,l}(z_1, \dots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \dots, (z_{n-1} - z_n)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$ such that the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$

satisfies the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,l}(z_1, \dots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \dots, n \tag{5.7}$$

in the region $|z_1| > \cdots > |z_n| > 0$.

Remark 5.8 Under the same condition as in the Theorem 5.5, it follows from the same argument in this section that matrix elements of iterates of logarithmic intertwining operators

$$\langle w'_{(0)}, \mathcal{Y}_1(\mathcal{Y}_2(w_1, z_1 - z_2), z_2) w_2 \rangle \tag{5.8}$$

also satisfy the expansions of the system of differential equations of the form (5.5) and (5.6) in the region $|z_2| > |z_1 - z_2| > 0$.

Example 5.9 Let V_L be the conformal vertex algebra associated with a nondegenerate even lattice L. Then any strongly graded generalized V_L-module W (in this example, all the generalized modules are modules) satisfies the assumption in Theorem 5.5 and the series (5.4), (5.8) satisfies the expansions of the system of differential equations (5.5) and (5.6) in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively.

6 The regularity of the singular points

We first recall the definition for *regular singular points* for a system of differential equations given in [K]. For the system of differential equations of form (5.7), *a singular point*

$$z_0 = (z_0^{(1)}, \dots, z_0^{(n)})$$

is an isolated singular point of the coefficient matrix

$$a_{k,l}(z_1, \dots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \dots, (z_{n-1} - z_n)^{-1}]$$

for $k = 1, \dots, m$ and $l = 1, \dots, n$. For $s = (s_1, \dots, s_n) \in \mathbb{Z}_+^n$, set

$$|s| = \sum_{i=0}^{n} s_i$$

and

$$(\log(z - z_0))^s = (\log(z_1 - z_0^{(1)}))^{s_1} \cdots (\log(z_n - z_0^{(n)}))^{s_n}.$$

For $t = (t^{(1)}, \dots, t^{(n)}) \in \mathbb{C}^n$, set

$$(z - z_0)^t = (z_1 - z_0^{(1)})^{t^{(1)}} \cdots (z_n - z_0^{(n)})^{t^{(n)}}.$$

A singular point z_0 for the system of differential equations of form (5.7) is *regular* if every solution in the punctured disc $(D^\times)^n$

$$0 < |z_i - z_0^{(i)}| < a_i$$

with some $a_i \in \mathbb{R}_+$ $(i = 1, \dots, n)$ is of the form

$$\varphi(z) = \sum_{i=1}^{r} \sum_{|m|<M} (z - z_0)^{t_i} (\log(z - z_0))^m f_{t_i,m}(z - z_0)$$

with $M, r \in \mathbb{Z}_+$ and each $f_{t_i,m}(z - z_0)$ holomorphic in $(D^\times)^n$. Theorem B.16 in [K] gives a sufficient condition for a singular point of a system of differential equations to be regular.

As in [H], for $r \in \mathbb{R}$, we define the R-modules $F_r^{(z_1=z_2)}(R)$, $F_r^{(z_1=z_2)}(T^{(\beta_1,\beta_2,\beta_3)})$ and $F_r^{(z_1=z_2)}(\widetilde{T}^{(\beta_1,\beta_2,\beta_3)})$, which provide filtration associated to the singular point $z_1 = z_2$ on R, R-modules $T^{(\beta_1,\beta_2,\beta_3)}$ and $\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}$, respectively.

For convenience, we shall use $\widetilde{\beta_0}$ to denote $\widetilde{\beta_1} + \widetilde{\beta_2} + \widetilde{\beta_3}$ for $\widetilde{\beta_i} \in \beta_i + A_0$ $(i = 1, 2, 3)$. We shall also consider the ring $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ and the $\mathbb{C}[z_1^{\pm}, z_2^{\pm}]$-module

$$(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)} = \coprod_{(\widetilde{\beta_1},\widetilde{\beta_2},\widetilde{\beta_3}) \in I^{(\beta_1,\beta_2,\beta_3)}} \mathbb{C}[z_1^{\pm}, z_2^{\pm}] \otimes W_0^{(\widetilde{\beta_0})} \otimes W_1^{(\widetilde{\beta_1})} \otimes W_2^{(\widetilde{\beta_2})} \otimes W_3^{(\widetilde{\beta_3})}.$$

Let $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}_{(r)}$ be the space of elements of $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$ of weight r for $r \in \mathbb{R}$. Let $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}) = \coprod_{s \le r} (T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}_{(s)}$. These subspaces give a filtration of $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$ in the following sense: $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}) \subset F_s((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$ for $r \le s$ and $(T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)} = \coprod_{r \in \mathbb{R}} F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$.

Let $F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) = F_r^{(z_1=z_2)}(\widetilde{T}^{(\beta_1,\beta_2,\beta_3)}) \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ for $r \in \mathbb{R}$. We have the following lemma:

Lemma 6.1 *For any $r \in \mathbb{R}$, $F_r((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}) \subset F_r^{(z_1=z_2)}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$.*

Proof. The proof is similar to the proof of Proposition 5.2 except some slight differences. We discuss elements of the form $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ with weight s, where $w_i \in W_i^{(\widetilde{\beta_i})}$ for $i = 0, 1, 2, 3$ and $u \in (V_0)_+$. By definition of the element $\mathcal{A}(u, w_0, w_1, w_2, w_3)$ in the R-submodule $\tilde{J}^{(\beta_1,\beta_2,\beta_3)}$, we have

$$\begin{aligned} w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3 & \\ = \sum_{k \ge 0} \binom{-1}{k} (-z_1)^k u^*_{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes w_3 - \mathcal{A}(u, w_0, w_1, w_2, w_3) & \\ - \sum_{k \ge 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 & \\ - \sum_{k \ge 0} \binom{-1}{k} (-z_1)^{-1-k} w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3. & \end{aligned}$$

We know from Lemma 5.1 that the elements $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ and $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3$ for $k \geq 0$ are either in $S^{(\beta_1,\beta_2,\beta_3)} \subset \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ or in $T^{(\beta_1,\beta_2,\beta_3)}$ with weights less than the weight of $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$.

In the first case, since elements of the form $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3$ are in $F^{(z_1=z_2)}_{s-k-1}(\tilde{J})$, $(-(z_1 - z_2))^{-1-k}w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F^{(z_1=z_2)}_s(\tilde{J})$. Thus in this case, $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ is an element of $F^{(z_1=z_2)}_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$.

In the second case, by induction assumption, $u^*_{-1-k}w_0 \otimes w_1 \otimes w_2 \otimes w_3$, $w_0 \otimes w_1 \otimes w_2 \otimes u_k w_3 \in F^{(z_1=z_2)}_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$ and $w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F^{(z_1=z_2)}_{s-k-1}(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Hence the element $(-(z_1-z_2))^{-1-k}w_0 \otimes w_1 \otimes u_k w_2 \otimes w_3 \in F^{(z_1=z_2)}_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Thus in this case, $w_0 \otimes u_{-1}w_1 \otimes w_2 \otimes w_3$ can be written as a sum of an element of $F^{(z_1=z_2)}_s(\tilde{J}^{(\beta_1,\beta_2,\beta_3)})$ and an element of $F_M(T^{(\beta_1,\beta_2,\beta_3)})$. ∎

Using Lemma 6.1, we get the following refinement of proposition 5.2:

Proposition 6.2 *For any $r \in \mathbb{R}$, $F^{(z_1=z_2)}_r(T^{(\beta_1,\beta_2,\beta_3)}) \subset F^{(z_1=z_2)}_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) + F_M(T^{(\beta_1,\beta_2,\beta_3)})$. In particular, $F^{(z_1=z_2)}_r(T^{(\beta_1,\beta_2,\beta_3)}) = F^{(z_1=z_2)}_r(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) \cap T^{(\beta_1,\beta_2,\beta_3)} + F_M(T^{(\beta_1,\beta_2,\beta_3)})$.*

Proof. It is a consequence of the decomposition:

$$F^{(z_1=z_2)}_r(T^{(\beta_1,\beta_2,\beta_3)}) = \coprod_{i=0}^{r} (z_1 - z_2)^{-i} F_{r-i}((T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)})$$

and Lemma 6.1. ∎

Let $w_i \in W_i^{(\widetilde{\beta}_i)}$ for $i = 0, 1, 2, 3$ and $(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3) \in I^{(\beta_1,\beta_2,\beta_3)}$. Then by Proposition 6.2,

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3 = \mathcal{W}_1 + \mathcal{W}_2$$

where $\mathcal{W}_1 \in F^{(z_1=z_2)}_\sigma(\tilde{J}^{(\beta_1,\beta_2,\beta_3)}) \cap T^{(\beta_1,\beta_2,\beta_3)} = F^{(z_1=z_2)}_\sigma(T^{(\beta_1,\beta_2,\beta_3)}) \cap \tilde{J}^{(\beta_1,\beta_2,\beta_3)}$ and $\mathcal{W}_2 \in F_M(T^{(\beta_1,\beta_2,\beta_3)})$. Using the same proof as Lemma 2.2 in [H], we have the following lemma:

Lemma 6.3 *For any $s \in [0, 1)$, there exist $S \in \mathbb{R}$ such that $s+S \in \mathbb{Z}_+$ and for any $w_i \in W_i$, $i = 0, 1, 2, 3$, satisfying $\sigma \in s + \mathbb{Z}$, $(z_1 - z_2)^{\sigma+S}\mathcal{W}_2 \in (T^{(\beta_1,\beta_2,\beta_3)})^{(z_1=z_2)}$.*

Theorem 6.4 *Let V be a strongly A-graded vertex algebra with a vertex subalgebra V_0 strongly graded with respect to $A_0 \subset A$. Suppose that every strongly graded V-module satisfies C_1-cofiniteness condition as a V_0-module. Also suppose that for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule*

$$N^{M_3^{(\widetilde{\beta}_1+\widetilde{\beta}_2)}}_{M_1^{(\widetilde{\beta}_1)} M_2^{(\widetilde{\beta}_2)}} \neq 0$$

for only finitely many pairs $(\widetilde{\beta}_1, \widetilde{\beta}_2) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$. Let W_i, $w_i \in W_i$ for $i = 0, 1, 2, 3, 4$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be the same as in Theorem 5.5. For any possible singular point of the

form $(z_1 = 0, z_2 = 0, z_1 = \infty, z_2 = \infty, z_1 = z_2)$, $z_1^{-1}(z_1 - z_2) = 0$, *or* $z_2^{-1}(z_1 - z_2) = 0$, *there exist*

$$a_k(z_1, z_2), b_l(z_1, z_2) \in \mathbb{C}[z_1^{\pm}, z_2^{\pm}, (z_1 - z_2)^{-1}]$$

for $k = 1, \dots, m$ *and* $l = 1, \dots, n$, *such that this singular point of the system (5.5) and (5.6) satisfied by (5.4) is regular.*

Proof. The proof is the same as the proof of Theorem 2.3 in [H] except that we use Proposition 6.2 and Lemma 6.3 here. ∎

We can prove the following theorem using the same method, so we omit the proof here.

Theorem 6.5 *For any set of possible singular points of the system (5.7) in Theorem 5.7 of the form* $z_i = 0$ *or* $z_i = \infty$ *for some* i *or* $z_i = z_j$ *for some* $i \neq j$, *the* $a_{k,l}(z_1, \dots, z_n)$ *in Theorem 5.7 can be chosen for* $k = 1, \dots, m$ *and* $l = 1, \dots, n$ *so that these singular points are regular.*

7 Braided tensor category structure

In the logarithmic tensor category theory developed in [HLZ], the convergence and expansion property for the logarithmic intertwining operators are needed in the construction of the associativity isomorphism. In this section, we will recall the definition of convergence and expansion property for products and iterates of logarithmic intertwining operators and then follow [HLZ] to give sufficient conditions for a category to have these properties.

Throughout this section, we will let $\mathcal{M}_{sg}$ (respectively, $\mathcal{GM}_{sg}$) denote the category of the strongly $\tilde{A}$-graded (respectively, generalized) V-modules. We are going to study the subcategory $\mathcal{C}$ of $\mathcal{M}_{sg}$ (respectively, $\mathcal{GM}_{sg}$) satisfying the following assumptions.

Assumption 7.1 We shall assume the following:

- A is an abelian group and $\tilde{A}$ is an abelian group containing A as a subgroup.
- V is a strongly A-graded conformal vertex algebra with a strongly $A_0 \subset A$-graded vertex subalgebra V_0 and V is an object of $\mathcal{C}$ as a V-module.
- All (generalized) V-modules are lower bounded, satisfy the C_1-cofiniteness condition as V_0-modules and for any two fixed elements β_1 and β_2 in $\tilde{A}$ and any triple of strongly graded generalized V-modules M_1, M_2 and M_3, the fusion rule

  $$N^{M_3^{(\widetilde{\beta_1}+\widetilde{\beta_2})}}_{M_1^{(\widetilde{\beta_1})} M_2^{(\widetilde{\beta_2})}} \neq 0$$

  for only finitely many pairs $(\widetilde{\beta_1}, \widetilde{\beta_2}) \in (\beta_1 + A_0) \times (\beta_2 + A_0)$.
- For any object of $\mathcal{C}$, the (generalized) weights are real numbers and in addition there exist $K \in \mathbb{Z}$ such that $(L(0) - L(0)_s)^K = 0$ on the generalized module.

- $\mathcal{C}$ is closed under images, under the contragredient functor, under taking finite direct sums.

Given objects W_1, W_2, W_3, W_4, M_1 and M_2 of the category $\mathcal{C}$, let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators of types $\binom{W_4}{W_1\, M_1}$, $\binom{M_1}{W_2\, W_3}$, $\binom{W_4}{M_2\, W_3}$ and $\binom{M_2}{W_1\, W_2}$, respectively. We recall the following definitions and theorems from Section 11 in [HLZ] (part VII):

Convergence and extension property for products For any $\beta \in \tilde{A}$, there exists an integer N_β depending only on $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and β, and for any doubly homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ ($\beta_1, \beta_2 \in \tilde{A}$) and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta,$$

there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \ldots, M$, and analytic functions $f_k(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + s_k > N_\beta,\ k = 1, \ldots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle_{W_4}|_{x_1=z_1,\ x_2=z_2}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1 - z_2)^{s_k}(\log z_2)^{i_k}(\log(z_1 - z_2))^{j_k} f_k(\frac{z_1 - z_2}{z_2})$$

(here $\log(z_1 - z_2)$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_2| > |z_1 - z_2| > 0$.

Convergence and extension property without logarithms for products When $i_k = j_k = 0$ for $k = 1, \ldots, M$, we call the property above the *convergence and extension property without logarithms for products*.

Convergence and extension property for iterates For any $\beta \in \tilde{A}$, there exists an integer $\tilde{N}_\beta$ depending only on $\mathcal{Y}^1$ and $\mathcal{Y}^2$ and β, and for any doubly homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ ($\beta_1, \beta_2 \in \tilde{A}$) and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta,$$

there exist $\tilde{M} \in \mathbb{N}$, $\tilde{r_k}, \tilde{s_k} \in \mathbb{R}$, $\tilde{i_k}, \tilde{j_k} \in \mathbb{N}$, $k = 1, \ldots, \tilde{M}$, and analytic functions $\tilde{f}_k(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\mathrm{wt}\ w_{(1)} + \mathrm{wt}\ w_{(2)} + \tilde{s_k} > \tilde{N}_\beta,\ k = 1, \ldots, \tilde{M},$$

such that

$$\langle w'_{(0)}, \mathcal{Y}_1(\mathcal{Y}_2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle_{W_4}|_{x_0=z_1-z_2,\ x_2=z_2}$$

is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{\tilde{M}} z_1^{\tilde{r_k}} z_2^{\tilde{s_k}} (\log z_1)^{\tilde{i_k}} (\log z_2)^{\tilde{j_k}} \tilde{f}_k(\frac{z_2}{z_1})$$

(here $\log z_1$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_1| > |z_2| > 0$.

Convergence and extension property without logarithmic for iterates When $i_k = j_k = 0$ for $k = 1, \dots, M$, we call the property above the *convergence and extension property without logarithms for iterates.*

If the convergence and extension property (with or without logarithms) for products holds for any objects W_1, W_2, W_3, W_4 and M_1 of $\mathcal{C}$ and any logarithmic intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the types as above, we say that *the convergence and extension property for products holds in $\mathcal{C}$.* We similarly define the meaning of the phrase *the convergence and extension property for iterates holds in $\mathcal{C}$.*

The following theorem generalizes Theorem 11.8 in [HLZ] to the strongly graded generalized modules for a strongly graded conformal vertex algebra:

Theorem 7.2 *Let V be a strongly graded conformal vertex algebra. Then*

1. *The convergence and extension properties for products and iterates hold in $\mathcal{C}$. If $\mathcal{C}$ is in $\mathcal{M}_{sg}$ and if every object of $\mathcal{C}$ is a direct sum of irreducible objects of $\mathcal{C}$ and there are only finitely many irreducible objects of $\mathcal{C}$ (up to equivalence), then the convergence and extension properties without logarithms for products and iterates hold in $\mathcal{C}$.*

2. *For any $n \in \mathbb{Z}_+$, any objects $W_1, \dots, W_{n+1}$ and $\widetilde{W_1}, \dots, \widetilde{W_{n-1}}$ of $\mathcal{C}$, any logarithmic intertwining operators*
$$\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$
of types
$$\binom{W_0}{W_1\, \widetilde{W_1}}, \binom{\widetilde{W_1}}{W_2\, \widetilde{W_2}}, \dots, \binom{\widetilde{W_{n-2}}}{W_{n-1}\, \widetilde{W_{n-1}}}, \binom{\widetilde{W_{n-1}}}{W_n\, W_{n+1}},$$
respectively, and any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \dots, W_{(n+1)} \in W_{n+1}$, the series
$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$
is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_1 \neq 0$, $i = 1, \dots, n$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0, z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points.

Proof. The first statement in the first part and the statement in the second part of the theorem follow directly from Theorem 5.7 and Theorem 6.5 and the theorem of differential equations with regular singular points. The second statement in the first part can be proved using the same method in [H]. ∎

In order to construct braided tensor category on the category of strongly graded generalized V-modules, we need the following assumption on $\mathcal{C}$ (see Assumption 10.1, Theorem 11.4 of [HLZ]).

Assumption 7.3 *Suppose the following two conditions are satisfied:*

1. *$\mathcal{C}$ is closed under $P(z)$-tensor products for some $z \in \mathbb{C}^{\times}$.*
2. *Every finite-generated lower bounded doubly graded generalized V-module is an object of $\mathcal{C}$.*

Conjecture 7.4 We conjectured that the category of certain strongly graded generalized V-modules satisfying the first condition in Assumption 7.3. The case for the vertex operator algebra was proved in [H1].

Under Assumption 7.1 and Assumption 7.3 on the category $\mathcal{C} \subset \mathcal{GM}_{sg}$, we generalize the main result (Theorem 12.15) of [HLZ] to the category of strongly graded generalized modules for a strongly graded vertex algebra:

Theorem 7.5 *Let V be a strongly graded conformal vertex algebra. Then the category $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r in [HLZ], is an additive braided tensor category.*

In the case that $\mathcal{C}$ is an abelian category, we have:

Corollary 7.6 *If the category $\mathcal{C}$ is an abelian category, then $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r in [HLZ], is a braided tensor category.*

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